Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series
Differential Calculus #8

The Chain Rule

Example 1:

The equation $x^2 + y^2 = 9$ gives a circle centred at the origin with radius 3.

We can solve for y as follows:

$$y^2 = 9 - x^2$$

So

$$y = \sqrt{(9 - x^2)} = (9 - x^2)^{\frac{1}{2}}$$

Here we have a chain of functions. We start with x then calculate $9 - x^2$ and finally take the square root. Spreading this out we have:

$$u = 9 - x^2$$

$$y = u^{\frac{1}{2}}$$

In this example, I have ignored the domains of the functions.

Remember: in real numbers, there is no square root for a negative number.

Here are some other examples of chains of functions. That just means we plug each one into the next.

Example 2.

$$y = (x^2 + 1)^4$$

so write

$$u = x^2 + 1 \text{ and } y = u^4$$

Example 3.

$$y = (\sin x)^{\frac{1}{2}}$$

so write

$$u = \sin x \text{ and } y = u^{\frac{1}{2}}$$

Example 4. NOT THE SAME AS EXAMPLE 3

$$y = sin(x^{\frac{1}{2}})$$

so write

$$u = x^{\frac{1}{2}} \text{ and } y = \sin u$$

The Chain Rule

If y can be written as a function $f(u)$ of u and then u can be written as a function $g(x)$ of x then the formula for the derivative is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example

If $y = (x^2 + 1)^4$ then

we can write $y = u^4$ with $u = x^2 + 1$.

So

$$\frac{dy}{du} = 4u^3$$

$$\frac{du}{dx} = 2x$$

and using the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot 2x = 4(x^2 + 1)^3(2x)$$

Example

If $y = (9 - x^2)^{\frac{1}{2}}$ then

write $y = u^{1/2}$ with $u = 9 - x^2$.

So

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx} = -2x$$

and using the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (-2x)$$

$$= \frac{1}{2}(9 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{9 - x^2}}$$

Sketch of the Proof of the Chain Rule

Let $y = f(g(x))$ and suppose we want the derivative at a.

$$\frac{f\big(g(x)\big) - f(g(a))}{x - a}$$

$$= \frac{f\big(g(x)\big) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}$$

As x goes to a also $g(x)$ will go to $g(a)$. This requires proof. It makes the first quotient into the quotient in the definition of a derivative.

Essentially, since $g(x)$ has a derivative at a, the limit

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

exists. Then since the denominator goes to 0, the numerator must also go to 0.

Summary

1. The Chain Rule is very important and easy to understand. With practice you can become proficient.

2. The Chain Rule calculates the derivative of a function that is obtained by substituting a series of functions one into the other.

3. If $y = f\,(g(x))$ then, with $u = g(x)$, we have $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df}{du} \cdot \frac{dg}{dx}$

4. With practice you will be able to drop the use of the “let $u = \ldots$ “ step

5. Chain Rule is the basis of the most powerful integration technique – substitution.